

One Dimensional Locally Connected S-spaces ^{*}

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Abstract

We construct, assuming Jensen's principle \diamond , a one-dimensional locally connected hereditarily separable continuum without convergent sequences.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. A *continuum* is any compact connected space. A *nontrivial convergent sequence* is a convergent ω -sequence of distinct points. As usual, $\dim(X)$ is the covering dimension of X ; for details, see Engelking [5]. “HS” abbreviates “hereditarily separable”. We shall prove:

Theorem 1.1 *Assuming \diamond , there is a locally connected HS continuum Z such that $\dim(Z) = 1$ and Z has no nontrivial convergent sequences.*

Note that points in Z must have uncountable character, so that Z is not hereditarily Lindelöf; thus, Z is an S-space.

Spaces with some of these features are well-known from the literature. A compact F-space has no nontrivial convergent sequences. Such a space can be a continuum; for example, the Čech remainder $\beta[0, 1) \setminus [0, 1)$ is connected, although not locally connected; more generally, no infinite compact F-space can be either locally connected or HS. In [13], van Mill constructs, under the Continuum Hypothesis, a locally connected continuum with no nontrivial convergent sequences. Van Mill's example,

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constructed as an inverse limit of Hilbert cubes, is infinite dimensional. Here, we shall replace the Hilbert cubes by one-dimensional Peano continua (i.e., connected, locally connected, compact metric spaces) to obtain a one-dimensional limit space. Our $Z = Z_{\omega_1}$ will be the limit of an inverse system $\langle Z_\alpha : \alpha < \omega_1 \rangle$. Each Z_α will be a copy of the *Menger sponge* [11] (or Menger curve) **MS**; this one-dimensional Peano continuum has homogeneity properties similar to those of the Hilbert cube. The basic properties of **MS** are summarized in Section 2, and Theorem 1.1 is proved in Section 3.

In [13], as well as in earlier work by Fedorchuk [7] and van Douwen and Fleissner [3], one kills all possible nontrivial convergent sequences in ω_1 steps. Here, we focus primarily on obtaining an S-space, modifying the construction of the original Fedorchuk S-space [6]; we follow the exposition in [4], where the lack of convergent sequences occurs only as an afterthought.

We do not know whether one can obtain Z so that it satisfies Theorem 1.1 with the stronger property $\text{ind}(Z) = 1$; that is, the open $U \subseteq Z$ with ∂U zero-dimensional form a base. In fact, we can easily modify our construction to ensure that $1 = \dim(Z) < \text{ind}(Z) = \infty$; this will hold because (as in [4]) we can give Z the additional property that all perfect subsets are G_δ sets; see Section 5 for details.

We can show that a Z satisfying Theorem 1.1 cannot have the property that the open $U \subseteq Z$ with ∂U scattered form a base; see Theorem 4.12 in Section 4. This strengthening of $\text{ind}(Z) = 1$ is satisfied by some well-known Peano continua. It is also satisfied by the space produced in [8] under \diamond by an inductive construction related to the one we describe here, but the space of [8] was not locally connected, and it had nontrivial convergent sequences (in fact, it was hereditarily Lindelöf).

2 On Sponges

The Menger sponge **MS** [11] is obtained by drilling holes through the cube $[0, 1]^3$, analogously to the way that one obtains the middle-third Cantor set by removing intervals from $[0, 1]$. The paper of Mayer, Oversteegen, and Tymchatyn [12] has a precise definition of **MS** and discusses its basic properties.

In proving theorems about **MS**, one often refers not to its definition, but to the following theorem of R. D. Anderson [1, 2] (or, see [12]), which characterizes **MS**. This theorem will be used to verify inductively that $Z_\alpha \cong \mathbf{MS}$. The fact that **MS** satisfies the stated conditions is easily seen from its definition, but it is not trivial to prove that they characterize **MS**.

Theorem 2.1 ***MS** is, up to homeomorphism, the only one-dimensional Peano continuum with no locally separating points and no non-empty planar open sets.*

Here, $C \subseteq X$ is *locally separating* iff, for some connected open $U \subseteq X$, the set $U \setminus C$ is not connected. A point x is locally separating iff $\{x\}$ is. This notion is applied in the Homeomorphism Extension Theorem of Mayer, Oversteegen, and Tymchatyn [12]:

Theorem 2.2 *Let K and L be closed, non-locally-separating subsets of \mathbf{MS} and let $h : K \rightarrow L$ be a homeomorphism. Then h extends to a homeomorphism of \mathbf{MS} onto itself.*

The non-locally-separating sets have the following closure property of Kline [9] (or, see Theorem 2.2 of [12]):

Theorem 2.3 *Let X be compact and locally connected, and let $K = \bigcup \{K_i : i \in \omega\}$, where K and the K_i are closed subsets of X . If K is locally separating then some K_i is locally separating.*

For example, these results imply that in \mathbf{MS} , all convergent sequences are equivalent. More precisely, points in \mathbf{MS} are not locally separating, so if $\langle x_i : i \in \omega \rangle$ converges to x_ω , then $\{x_i : i \leq \omega\}$ is not locally separating. Thus, if $\langle s_i \rangle$ and $\langle t_i \rangle$ are nontrivial convergent sequences in \mathbf{MS} , with limit points s_ω and t_ω , respectively, then there is a homeomorphism of \mathbf{MS} onto itself that maps s_i to t_i for each $i \leq \omega$.

The following consequence of Theorem 2.1 was noted by Prajs [14] (see p. 657).

Lemma 2.4 *Let $J \subseteq \mathbf{MS}$ be a non-locally-separating arc and obtain \mathbf{MS}/J by collapsing J to a point. Then $\mathbf{MS}/J \cong \mathbf{MS}$ and the natural map $\pi : \mathbf{MS} \rightarrow \mathbf{MS}/J$ is monotone.*

Here, a map $f : Y \rightarrow X$ is called *monotone* iff each $f^{-1}\{x\}$ is connected; so, the monotonicity in Lemma 2.4 is obvious. When X, Y are compact, monotonicity implies that $f^{-1}(U)$ is connected whenever U is a connected open or closed subset of X .

We shall use these results to show that the property of being a Menger sponge will be preserved at the limit stages of our construction:

Lemma 2.5 *Suppose that γ is a countable limit ordinal and Z_γ is an inverse limit of $\langle Z_\alpha : \alpha < \gamma \rangle$, where all bonding maps π_α^β are monotone and each $Z_\alpha \cong \mathbf{MS}$. Then $Z_\gamma \cong \mathbf{MS}$.*

Proof. We verify the conditions of Theorem 2.1. $\dim(Z_\gamma) = 1$, since this property is preserved by inverse limits of compacta, and Z_γ is locally connected because the π_α^β are monotone. So, we need to verify that Z_γ has no locally separating points and no non-empty planar open sets.

Suppose that $q \in Z_\gamma$ is locally separating; so we have a connected neighborhood U of q with $U \setminus \{q\}$ not connected. Shrinking U , we may assume that $U = (\pi_\alpha^\gamma)^{-1}(V)$, where $\alpha < \gamma$ and V is open and connected in Z_α . Since $Z_\alpha \cong \mathbf{MS}$, $\pi_\alpha^\gamma(q)$ is not locally separating, so $V \setminus \{\pi_\alpha^\gamma(q)\}$ is connected. Then, since π_α^γ is monotone, $(\pi_\alpha^\gamma)^{-1}(V \setminus \{\pi_\alpha^\gamma(q)\}) = U \setminus (\pi_\alpha^\gamma)^{-1}\{\pi_\alpha^\gamma(q)\}$ is connected. The same argument shows that $U \setminus (\pi_\beta^\gamma)^{-1}\{\pi_\beta^\gamma(q)\}$ is connected whenever $\alpha \leq \beta < \gamma$. But then $U \setminus \{q\} = \bigcup \{U \setminus (\pi_\beta^\gamma)^{-1}\{\pi_\beta^\gamma(q)\} : \alpha \leq \beta < \gamma\}$ is connected also.

Suppose that $U \subseteq Z_\gamma$ is open and non-empty; we show that U is not planar. Shrinking U , we may assume that $U = (\pi_\alpha^\gamma)^{-1}(V)$, where $\alpha < \gamma$ and V is open in Z_α . Since $Z_\alpha \cong \mathbf{MS}$, there is a K_5 set $F \subseteq V$; that is, F consists of 5 distinct points p_0, p_1, p_2, p_3, p_4 together with arcs $J_{i,j}$ with endpoints p_i, p_j for $0 \leq i < j < 5$, where the sets $J_{i,j} \setminus \{p_i, p_j\}$, for $0 \leq i < j < 5$, are pairwise disjoint. Now F is not planar, and, one can show that $(\pi_\alpha^\gamma)^{-1}(F)$ is not planar either. To do this, use the fact that π_α^γ is monotone, so that the sets $(\pi_\alpha^\gamma)^{-1}\{p_i\}$ and $(\pi_\alpha^\gamma)^{-1}(J_{i,j})$ are all continua. ☺

The following terminology was used also in the exposition in [4] of the Fedorchuk S-space:

Definition 2.6 *Let \mathcal{F} be a family of subsets of X . Then $x \in X$ is a strong limit point of \mathcal{F} iff for all neighborhoods U of x , there is an $F \in \mathcal{F}$ such that $F \subseteq U$ and $x \notin F$.*

In practice, we shall only use this notion when the elements of \mathcal{F} are closed. If all elements of \mathcal{F} are singletons, this reduces to the usual notion of a point being a limit point of a set of points.

The map $\pi_{\alpha+1}^{\alpha+1} : Z_{\alpha+1} \rightarrow Z_\alpha$ will always be obtained by collapsing a non-locally-separating arc in $Z_{\alpha+1}$ to a point. We obtain it using:

Lemma 2.7 *Assume that $X \cong \mathbf{MS}$ and that for $n \in \omega$, \mathcal{F}_n is a family of non-locally-separating closed subsets of X . Fix $t \in X$ such that t is a strong limit point of each \mathcal{F}_n . Then there is a $Y \cong \mathbf{MS}$ and a monotone $\pi : Y \rightarrow X$ such that $\pi^{-1}\{t\}$ is a non-locally-separating arc in Y , $|\pi^{-1}\{x\}| = 1$ for all $x \neq t$, and, for each n and each $y \in \pi^{-1}\{t\}$: y is a strong limit point of $\{\pi^{-1}(F) : F \in \mathcal{F}_n\}$.*

Proof. First, let $\{A_n : n \in \omega\}$ partition ω into disjoint infinite sets. In X , choose disjoint closed $F_i \not\ni t$ for $i \in \omega$ such that $F_i \in \mathcal{F}_n$ whenever $i \in A_n$, and such that every neighborhood of t contains all but finitely many of the F_i . Let $L = \{t\} \cup \bigcup_i F_i$. Then L is closed and non-locally-separating by Theorem 2.3.

Now, in \mathbf{MS} , let J be any non-locally-separating arc. Choose disjoint closed non-locally separating sets G_i for $i \in \omega$ such that each $G_i \cong F_i$, every neighborhood of J contains all but finitely many G_i , each $G_i \cap J = \emptyset$, and for each n and each $y \in J$: y is a strong limit point of $\{G_i : i \in A_n\}$.

Let $\sigma : \mathbf{MS} \twoheadrightarrow \mathbf{MS}/J$ be the usual projection, and let $[J]$ denote the point to which σ collapses the set J . Then $\mathbf{MS}/J \cong \mathbf{MS}$ by Lemma 2.4. In \mathbf{MS}/J , let $K = \{[J]\} \cup \bigcup \{\sigma(G_i) : i \in \omega\}$. Let $h : K \twoheadrightarrow L$ be a homeomorphism such that $h([J]) = t$ and each $h(\sigma(G_i)) = F_i$. By Theorem 2.2, h extends to a homeomorphism $\tilde{h} : \mathbf{MS}/J \twoheadrightarrow X$.

Now, let $Y = \mathbf{MS}$ and let $\pi = \tilde{h} \circ \sigma$. ☺

The next lemma will simplify somewhat the description of our inverse limit:

Lemma 2.8 *In Lemma 2.7, we may obtain $Y \subseteq X \times [0, 1]$, with $\pi : Y \twoheadrightarrow X$ the natural projection.*

Proof. Start with any Y, π, t satisfying Lemma 2.7, and let $J := \pi^{-1}\{t\}$. Apply the Tietze Extension Theorem to fix $f : Y \twoheadrightarrow [0, 1]$ such that $f|_J : J \twoheadrightarrow [0, 1]$ is a homeomorphism. Then $y \mapsto (\pi(y), f(y))$ is one-to-one on Y , and hence $\tilde{Y} := \{(\pi(y), f(y)) : y \in Y\} \subseteq X \times [0, 1]$ satisfies Lemma 2.8. ☺

The following additional property of our π will be useful:

Lemma 2.9 *Let t and $\pi : Y \twoheadrightarrow X$ be as in Lemma 2.7 or 2.8. Assume that $H \subseteq X$ is closed and nowhere dense and not locally separating. Then $\pi^{-1}(H) \subseteq Y$ is closed and nowhere dense and not locally separating.*

Proof. $\pi^{-1}(H)$ is closed and nowhere dense because π is continuous and irreducible. Also note that $\pi^{-1}(H)$ is not locally separating if either $H = \{t\}$ (trivially) or $t \notin H$ (because π is a homeomorphism in a neighborhood of $\pi^{-1}(H)$).

Next, note that every closed $K \subseteq H$ is non-locally-separating in X : If not, let $U \subseteq X$ be connected and open with $U \setminus K$ not connected, so that $U \setminus K = W_0 \cup W_1$, where the W_i are open in X , non-empty, and disjoint. Then $U \setminus H = W_0 \setminus H \cup W_1 \setminus H$, but H is not locally separating, so one of the $W_i \setminus H = \emptyset$, so $W_i \subseteq H$, contradicting H being nowhere dense.

Now, let $H = \bigcup_{n \in \omega} K_n$, where each K_n is closed and either $K_n = \{t\}$ or $t \notin K_n$. Then $\pi^{-1}(H) = \bigcup_n \pi^{-1}(K_n)$, which is not locally separating by Theorem 2.3. ☺

3 The Inverse Limit

We shall obtain our space $Z = Z_{\omega_1}$ as an inverse limit of a sequence $\langle Z_\alpha : \alpha < \omega_1 \rangle$. As with many such constructions, it is somewhat simpler to view the Z_α concretely as subsets of cubes, so that the bonding maps are just projections. Thus, we shall have:

Conditions 3.1 We obtain Z_α for $\alpha \leq \omega_1$ and $\pi_\alpha^\beta, \sigma_\alpha^\beta$ for $\alpha \leq \beta \leq \omega_1$ such that:

- C1. Each Z_α is a closed subset of $\mathbf{MS} \times [0, 1]^\alpha$, and $Z_0 = \mathbf{MS}$.
- C2. For $\alpha \leq \beta \leq \omega_1$, $\pi_\alpha^\beta : \mathbf{MS} \times [0, 1]^\beta \rightarrow \mathbf{MS} \times [0, 1]^\alpha$ is the natural projection.
- C3. $\pi_\alpha^\beta(Z_\beta) = Z_\alpha$ whenever $\alpha \leq \beta \leq \omega_1$.
- C4. Z_α is homeomorphic to \mathbf{MS} whenever $\alpha < \omega_1$.
- C5. The maps $\sigma_\alpha^\beta := \pi_\alpha^\beta \upharpoonright Z_\beta : Z_\beta \rightarrow Z_\alpha$, for $\alpha \leq \beta \leq \omega_1$, are monotone.

Using (C1,C2,C3), the construction is determined at limit ordinals; (C4) is preserved by Lemma 2.5 and (C5). It remains to explain how, given Z_α for $\alpha < \omega_1$, we obtain $Z_{\alpha+1} \subseteq Z_\alpha \times [0, 1]$; as usual, we identify $\mathbf{MS} \times [0, 1]^{\alpha+1}$ with $\mathbf{MS} \times [0, 1]^\alpha \times [0, 1]$.

We now add:

Conditions 3.2 We have q_α^ξ and t_α for $\xi < \alpha < \omega_1$ such that:

- C6. Each $\langle q_\alpha^\xi : \xi < \alpha \rangle$ is a sequence of points in $\mathbf{MS} \times [0, 1]^\alpha$.
- C7. Whenever $\langle q^\xi : \xi < \omega_1 \rangle$ is any sequence of points in $\mathbf{MS} \times [0, 1]^{\omega_1}$, $\{\alpha < \omega_1 : \forall \xi < \alpha [\pi_\alpha^{\omega_1}(q^\xi) = q_\alpha^\xi]\}$ is stationary.
- C8. Whenever $\alpha < \beta \leq \omega_1$ and $z \in Z_\alpha$: If $q_\alpha^\xi \in Z_\alpha$ for all $\xi < \alpha$ and z is a limit point of $\langle q_\alpha^\xi : \xi < \alpha \rangle$, then all points of $(\sigma_\alpha^\beta)^{-1}\{z\}$ are strong limit points of $\{(\sigma_\alpha^\beta)^{-1}\{q_\alpha^\xi\} : \xi < \alpha\}$.
- C9. $t_\alpha \in Z_\alpha$, and for all $z \in Z_\alpha$: $(\sigma_\alpha^{\alpha+1})^{-1}\{z\}$ is a singleton if $z \neq t_\alpha$ and a non-locally-separating arc if $z = t_\alpha$.
- C10. $t_\alpha = q_\alpha^0$ whenever $\alpha > 0$ and $q_\alpha^0 \in Z_\alpha$.

Proof of Theorem 1.1. The fact that one may obtain (C1 – C10) has already been outlined above. (C6,C7) are possible by \diamond , and (C10) is just a definition. (C8,C9) are obtained by induction on β . For the successor step, we must obtain $Z_{\beta+1}$ from Z_β using Lemmas 2.7 and 2.8. Here, $X = Z_\beta$, $Y = Z_{\beta+1}$, and $t = t_\beta$; the \mathcal{F}_n list all sets of the form $\mathcal{F}_\alpha^\beta := \{(\sigma_\alpha^\beta)^{-1}\{q_\alpha^\xi\} : \xi < \alpha \text{ \& } q_\alpha^\xi \in Z_\alpha\}$ such that $\alpha \leq \beta$ and t_β is a strong limit point of \mathcal{F}_α^β . Observe that (C8) for $(\alpha, \beta + 1)$ is immediate from (C8) for (α, β) *except* for the points of $Z_{\beta+1}$ in $(\sigma_\beta^{\beta+1})^{-1}\{t_\beta\}$. Also observe that in order to apply Lemmas 2.7 and 2.8, we must check by induction on β , using Lemma 2.9, that the sets $(\sigma_\alpha^\beta)^{-1}\{q_\alpha^\xi\}$ are non-locally-separating (and nowhere dense) in Z_β .

Note that $\chi(z, Z) = \aleph_1$ for all $z \in Z$; this follows from (C9,C10) and the fact, using (C7), that $\{\alpha < \omega_1 : \pi_\alpha^{\omega_1}(z) = t_\alpha\}$ is unbounded in ω_1 .

Z is HS by (C6,C7,C8,C1,C2,C3): If not, suppose that $\langle q^\xi : \xi < \omega_1 \rangle$ is left-separated in Z . As in [4], we get a club $C \subset \omega_1$ such that for all $\alpha \in C$,

$$\text{cl} \{ \sigma_\alpha^{\omega_1}(q^\xi) : \xi < \alpha \} \supseteq \{ \sigma_\alpha^{\omega_1}(q^\xi) : \xi < \omega_1 \} \quad .$$

Fix $\alpha \in C$ such that $\forall \xi < \alpha [\sigma_\alpha^{\omega_1}(q^\xi) = q_\alpha^\xi]$. Let $z = \sigma_\alpha^{\omega_1}(q^\alpha)$. Applying (C8) with $\beta = \omega_1$, we have in Z : all points of $(\sigma_\alpha^{\omega_1})^{-1}\{z\}$ are strong limit points of $\{(\sigma_\alpha^{\omega_1})^{-1}\{q_\alpha^\xi\} : \xi < \alpha\}$. In particular, q^α is a limit point of $\langle q^\xi : \xi < \alpha \rangle$, contradicting “left-separated”.

Similarly, Z has no non-trivial convergent sequences: Suppose that $q^n \rightarrow q^\omega$ in Z , where the q^ξ for $\xi \leq \omega$ are distinct. Let $q^\xi = q^\omega$ when $\omega < \xi < \omega_1$, and apply (C7) to get α with $\omega < \alpha < \omega_1$ such that the $\sigma_\alpha^{\omega_1}(q^\xi)$ for $\xi \leq \omega$ are distinct points and $\forall \xi < \alpha [\sigma_\alpha^{\omega_1}(q^\xi) = q_\alpha^\xi]$. Let $z = \sigma_\alpha^{\omega_1}(q^\omega)$. Then all points of $(\sigma_\alpha^{\omega_1})^{-1}\{z\}$ are strong limit points of $\{(\sigma_\alpha^{\omega_1})^{-1}\{q_\alpha^\xi\} : \xi < \alpha\}$ and hence also of $\{(\sigma_\alpha^{\omega_1})^{-1}\{q_\alpha^n\} : n < \omega\}$. So, all points of $(\sigma_\alpha^{\omega_1})^{-1}\{z\}$ are limit points of $\{q^n : n \in \omega\}$. Since $\{q^\omega\} \subsetneq (\sigma_\alpha^{\omega_1})^{-1}\{z\}$ (by $\chi(q^\omega, Z) = \aleph_1$), we contradict $q^n \rightarrow q^\omega$. ☺

4 The Almost Clopen Algebra

We show here (Theorem 4.12) that a space Z satisfying Theorem 1.1 cannot have a base of open sets with scattered boundaries; equivalently (because there are no non-trivial convergent sequences) with finite boundaries. We first note that if there were such a base, we could take the basic open sets U to be regular, since $\partial(\text{int}(\text{cl}(U))) \subseteq \partial U$. To simplify notation, we define:

Definition 4.1 $\text{ro}(X)$ denotes the algebra of regular open subsets of X , and $\text{acl}(X)$ (the almost clopen sets) denotes the family of regular open sets U such that ∂U is finite. For $U \in \text{ro}(X)$, let U^c denote the boolean complement $(X \setminus U)^\circ$.

Note that $\text{acl}(X)$ is a boolean subalgebra of $\text{ro}(X)$: If $U \in \text{acl}(X)$ and $W = U^c$, then $\partial W = \partial U$, so $W \in \text{acl}(X)$. Also, if $U, V \in \text{acl}(X)$ and $W = U \wedge V = U \cap V \in \text{ro}(X)$, then $W \in \text{acl}(X)$ because $\partial(W) \subseteq \partial(U) \cup \partial(V)$.

In a locally connected space, the connected components of an open set U are open; if V is any such component, then $\partial V \subseteq \partial U$ (because V is relatively clopen in U), so $V \in \text{acl}(X)$ whenever $U \in \text{acl}(X)$. Thus,

Lemma 4.2 *If X is locally connected and $\text{acl}(X)$ is a local base at $p \in X$, then $\{U \in \text{acl}(X) : p \in U \text{ \& } U \text{ is connected}\}$ is also a local base at p .*

Various LOTS sums have bases of almost clopen sets. This is true, for example, for any compact hedgehog consisting of a central point plus arbitrarily many LOTS spines. The assumption of no convergent sequences, however, puts some restrictions on the space. In particular, the hedgehog fails the following lemma (taking U to be X and letting s be the central point):

Lemma 4.3 *Assume that X is compact and locally connected, and X has no nontrivial convergent sequences. Fix an open U with ∂U finite, and fix a finite $s \subseteq U$. Then $U \setminus s$ has finitely many components.*

Proof. Assume that V_n , for $n < \omega$, are different components of $U \setminus s$. Choose $x_n \in V_n$. Then the limit points of $\{x_n : n \in \omega\}$ must lie in $\partial(U \setminus s) \subseteq \partial U \cup s$. Thus, $\{x_n : n \in \omega\}$ has finitely many limit points, which is impossible if X has no nontrivial convergent sequences. ☺

We now look more closely at the locally separating points; that is, the points $p \in X$ such that $U \setminus \{p\}$ is not connected for some open connected $U \ni p$.

Definition 4.4 *If $p \in U \subseteq X$, then $c(p, U)$ is the number of components of $U \setminus \{p\}$.*

Lemma 4.5 *Assume that X is compact and locally connected. Fix $p \in X$ and open connected U, V with $p \in U \subseteq V$. Then:*

1. *Every component of $V \setminus \{p\}$ is a subset of exactly one component of $U \setminus \{p\}$.*
2. *If $V \subseteq U$, then $c(p, V) \geq c(p, U)$.*
3. *If $\text{acl}(X)$ is a local base at p and X has no nontrivial convergent sequences, then $c(p, U)$ is finite.*

Proof. (1) is immediate from the fact that if W is a component of $V \setminus \{p\}$ then W is connected and $W \subseteq U \setminus \{p\}$. For (2), use the fact that every component of $U \setminus \{p\}$ must meet V because U is connected, so that (1) provides a map from the components of $V \setminus \{p\}$ onto the components of $U \setminus \{p\}$. For (3), choose $V \in \text{acl}(X)$ with $p \in V \subseteq U$, and apply (2) and Lemma 4.3. ☺

The next lemma is trivial, but useful when ∂U is finite.

Lemma 4.6 *Suppose that $E \subseteq X$ is connected, $U \subseteq X$ is open, and $\partial U \cap E = \emptyset$. Then $E \subseteq U$ or $E \cap U = \emptyset$.*

Proof. $U \cap E = \overline{U} \cap E$ is relatively clopen in E , so $U \cap E$ is either E or \emptyset . ☺

Lemma 4.7 *Assume that X is compact and locally connected, $\text{acl}(X)$ is a local base at $p \in X$, and X has no nontrivial convergent sequences. Then there is an $n \in \omega$ such that $c(p, U) \leq n$ for all open connected $U \ni p$.*

Proof. If this fails, then applying Lemma 4.5, we may fix open connected $U_n \ni p$ for $n \in \omega$ such that $U_0 \supseteq \overline{U}_1 \supseteq U_1 \supseteq \overline{U}_2 \cdots$ and $2 \leq c(p, U_0) < c(p, U_1) < \cdots$. Then, we may define a subtree $T \subseteq \omega^{<\omega}$ and open connected W_s for $s \in T$ and $k_s \in \omega \setminus \{0\}$ for $s \in T$ as follows:

1. W_\emptyset is the component of p in X .
2. If $\text{lh}(s) = n$, then k_s is the number of components of $U_n \setminus \{p\}$ which are subsets of W_s , and these components are listed as $\{W_{s \smallfrown i} : i < k_s\}$.
3. $s \smallfrown i \in T$ iff $s \in T$ and $i < k_s$.

Item (1) is a bit artificial, but it gives T a root node (\cdot) . For the levels below the root, note that $|T \cap \omega^{n+1}| = c(p, U_n)$, and the W_s for $s \in T \cap \omega^{n+1}$ list the components of $U_n \setminus \{p\}$. Let $P(T) = \{f \in \omega^\omega : \forall n [f \upharpoonright n \in T]\}$ be the set of paths through T . Since every node in T has at least one child, $|P(T)|$ is either \aleph_0 or 2^{\aleph_0} . Note that $\text{cl}(W_{s \smallfrown i}) \subseteq W_s \cup \{p\}$, since if $n = \text{lh}(s) > 0$ and $q \in \text{cl}(W_{s \smallfrown i}) \setminus \{p\}$, then q and the points of $W_{s \smallfrown i}$ must all lie in the same component of $U_{n-1} \setminus \{p\}$, which is W_s .

Let $H = \bigcap_n U_n = \bigcap_n \overline{U_n}$. Then H is a connected closed G_δ containing p , and H must be infinite, since p must have uncountable character. For each $f \in P(T)$, let $K_f = \bigcap_n \text{cl}(W_{f \upharpoonright n}) = \{p\} \cup \bigcap_n W_{f \upharpoonright n}$. Then the K_f are connected and infinite, since $\{p\}$ cannot be a decreasing intersection of ω infinite closed sets (or there would be a convergent sequence). Observe that $K_f \cap K_g = \{p\}$ whenever $f \neq g$. Thus, if $p \in V \in \text{acl}(X)$ then $K_f \subseteq V$ for all but finitely many $f \in P(T)$, since $K_f \subseteq V$ whenever $K_f \cap \partial V = \emptyset$ by Lemma 4.6. Now let f_i , for $i \in \omega$ be distinct elements of $P(T)$, and choose $q_i \in K_{f_i} \setminus \{p\}$. Then every neighborhood of p contains all but finitely many q_i , so the q_i converge to p , a contradiction. ☺

Definition 4.8 *Assume that X is compact and locally connected, $\text{acl}(X)$ is a base for X , and X has no nontrivial convergent sequences. Then for each $p \in X$, define $c(p) \in \omega$ to be the largest $c(p, U)$ among all open connected $U \ni p$.*

By a standard chaining argument:

Lemma 4.9 *Assume that X is compact and locally connected and $\text{acl}(X)$ is a base for X . Fix a connected open $U \subseteq X$ and a compact $F \subseteq U$. Then there is a connected $V \in \text{acl}(X)$ such that $F \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. Let $\mathcal{G} = \{W \in \text{acl}(X) : \emptyset \neq \overline{W} \subseteq U \text{ \& } W \text{ is connected}\}$. Then $\bigcup \mathcal{G} = U$. View \mathcal{G} as an undirected graph, by putting an edge between W_1 and W_2 iff $W_1 \cap W_2 \neq \emptyset$. Then \mathcal{G} is connected as a graph because U is connected and the components of \mathcal{G} yield topological components of U . Fix a finite $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $F \subseteq \bigcup \mathcal{G}_0$. Then fix a finite connected \mathcal{G}_1 with $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}$. Let $V = \bigvee \mathcal{G}_1 = \text{int}(\text{cl}(\bigcup \mathcal{G}_1))$. ☺

Lemma 4.10 *Assume that X is compact and locally connected, $\text{acl}(X)$ is a base for X , and X has no nontrivial convergent sequences. Then there is no sequence of open sets $\langle U_n : n \in \omega \rangle$ such that $\overline{U_{n+1}} \subsetneq U_n$ for all n and $\overline{U_n} \setminus U_{n+1}$ is connected for all even n .*

Proof. Given such a sequence, choose $x_n \in \overline{U}_n \setminus U_{n+1}$, and let y be a limit point of $\{x_{2m} : m \in \omega\}$. Since $\langle x_{2m} : m \in \omega \rangle$ cannot converge to y , fix a connected $W \in \mathbf{acl}(X)$ and disjoint infinite $A, B \subseteq \{2m : m \in \omega\}$ such that $x_n \in W$ for all $n \in A$ and $x_n \notin W$ for all $n \in B$. Since ∂W is finite, we may also assume (shrinking A, B if necessary) that $\partial W \cap (\overline{U}_n \setminus U_{n+1}) = \emptyset$ for all $n \in A \cup B$. Then, by Lemma 4.6, $\overline{U}_n \setminus U_{n+1} \subseteq W$ for all $n \in A$ and $(\overline{U}_n \setminus U_{n+1}) \cap W = \emptyset$ for all $n \in B$. But then, for $n \in B$, the connected W is partitioned into the disjoint open sets $W \cap U_{n+1}$, $W \setminus \overline{U}_n$, both of which are non-empty when $n > \min(A)$. \odot

Lemma 4.11 *Assume that X is compact and locally connected, $\mathbf{acl}(X)$ is a base for X , and X has no nontrivial convergent sequences. Then every non-isolated point in X is locally separating.*

Proof. Suppose we have a non-isolated p which is not locally separating; so $U \setminus \{p\}$ is connected whenever U is open and connected. Then inductively construct U_n for $n \in \omega$ such that

1. Each U_n is open and $p \in U_n$.
2. Each $\overline{U}_{n+1} \subsetneq U_n$.
3. $\overline{U}_n \setminus U_{n+1}$ is connected whenever n is even.
4. Each $U_n \in \mathbf{acl}(X)$.
5. U_n is connected for all even n .

Then (1)(2)(3) contradict Lemma 4.10.

To construct the U_n : Let $U_0 \in \mathbf{acl}(X)$ be such that $p \in U_0$ and U_0 is connected and not clopen. Given U_n , where n is even, we construct U_{n+1} and U_{n+2} as follows:

Say $\partial U_n = \{q^j : j < r\}$; of course, r and the q^j depend on n . For each j , choose $V^j \in \mathbf{acl}(X)$ be such that $q^j \in V^j$, $p \notin \text{cl}(V^j)$, and V^j is connected. Also make sure that the \overline{V}^j are disjoint; then $\overline{V}^j \cap \partial U_n = \{q^j\}$. Let $\{W_i^j : i < c^j\}$ list the components of $V^j \setminus \{q^j\}$; so $2 \leq c^j < \omega$. Then W_i^j is connected and $\partial U_n \cap W_i^j = \emptyset$, so $W_i^j \subseteq U_n$ or $W_i^j \cap U_n = \emptyset$; say $W_i^j \subseteq U_n$ for $i < d^j$ and $W_i^j \cap U_n = \emptyset$ for $d^j \leq i < c^j$; so $1 \leq d^j < c^j$. Choose $y_i^j \in W_i^j$. Now U_n is connected and p is not locally separating, so $U_n \setminus \{p\}$ is connected. Applying Lemma 4.9, fix a connected $R \in \mathbf{acl}(X)$ such that $\{y_i^j : j < r \text{ \& } i < d^j\} \subseteq R \subseteq \overline{R} \subseteq U \setminus \{p\}$. Let S be the finite union $R \cup \bigcup \{W_i^j : j < r \text{ \& } i < d^j\}$. Then S is open and connected, $p \notin \overline{S}$, and each $q^j \in \overline{S}$. Let $U_{n+1} = U_n \setminus \overline{S} = \overline{U}_n \setminus \overline{S}$. Then $p \in U_{n+1} \in \mathbf{acl}(X)$, and $\overline{U}_n \setminus U_{n+1} = \overline{S}$ is connected. Also, each $q^j \notin \overline{U}_{n+1}$ because $U_{n+1} \cap V^j = \emptyset$, so that $\overline{U}_{n+1} \subseteq U_n$.

Now, choose a connected $U_{n+2} \in \mathbf{acl}(X)$ so that $p \in U_{n+2} \subseteq \overline{U}_{n+2} \subsetneq U_{n+1}$. \odot

Theorem 4.12 *If X is infinite, compact, locally connected, and $\mathbf{acl}(X)$ is a base for X , then X has a nontrivial convergent sequence.*

Proof. Suppose not. Fix any non-isolated $p \in X$; then p is locally separating by Lemma 4.11, so $c(p) \geq 2$ (see Definition 4.8). Fix a connected $U \in \mathbf{acl}(X)$ such that $p \in U$ and $c(p, U) = c(p)$. Let W_i , for $i < c(p)$ be the components of $U \setminus \{p\}$. Then $c(p, V) = c(p)$ whenever $V \in \mathbf{acl}(X)$ and $p \in V \subseteq U$; furthermore, the components of $V \setminus \{p\}$ are the sets $W_i \cap V$ for $i < c(p)$.

Let $Y = \text{cl}(W_0)$. Then $\mathbf{acl}(Y)$ is a base for Y , Y is locally connected, and Y has no nontrivial convergent sequences. Furthermore, $p \in Y$ and p is not locally separating in Y , contradicting Lemma 4.11 applied to Y . ☺

5 Further Remarks

We note that in constructing a locally connected compactum, the monotone bonding maps, as used also by van Mill [13], are inevitable:

Remark 5.1 *Assume that $X \subseteq [0, 1]^{\omega_1}$ is compact and locally connected. Define $X_\alpha = \pi_\alpha^{\omega_1}(X) \subseteq [0, 1]^\alpha$. Then there is a club $C \subseteq \omega_1$ such that X_α is locally connected for all $\alpha \in C$, and such that $\sigma_\alpha^\beta := \pi_\alpha^\beta \upharpoonright X_\beta$ is monotone whenever $\alpha < \beta$ and $\alpha, \beta \in C \cup \{\omega_1\}$.*

Proof. Let \mathcal{B} be the family of all connected open F_σ subsets of X . Then \mathcal{B} is a base for X . For $\alpha < \omega_1$, let \mathcal{B}_α be the family of all open $U \subseteq X_\alpha$ such that $(\sigma_\alpha^{\omega_1})^{-1}(U) \in \mathcal{B}$. Observe that each $U \in \mathcal{B}_\alpha$ is connected. Put $\alpha \in C$ iff \mathcal{B}_α is a base for X_α . Then C is club.

Now, it is sufficient to show that $(\sigma_\alpha^{\omega_1})^{-1}\{x\}$ is connected whenever $\alpha \in C$ and $x \in X_\alpha$. Choose $U_n \in \mathcal{B}_\alpha$ with $x \in U_n \supseteq \overline{U_{n+1}}$ for all $n \in \omega$ and $\{x\} = \bigcap_n U_n = \bigcap_n \overline{U_n}$. Each $(\sigma_\alpha^{\omega_1})^{-1}(U_n)$ is in \mathcal{B} , so it and its closure are connected, and $\text{cl}((\sigma_\alpha^{\omega_1})^{-1}(U_{n+1})) \subseteq (\sigma_\alpha^{\omega_1})^{-1}(\overline{U_{n+1}}) \subseteq (\sigma_\alpha^{\omega_1})^{-1}(U_n)$, so that $(\sigma_\alpha^{\omega_1})^{-1}\{x\}$ is the decreasing intersection of the connected closed sets $\text{cl}((\sigma_\alpha^{\omega_1})^{-1}(U_n))$, and is hence connected. ☺

We do not know if conditions (C1 – C10) in Section 3 determine $\text{ind}(Z)$, but a minor addition to the construction will ensure that Z does not have small *transfinite inductive dimension*; that is, $\text{trind}(Z) = \infty$ (and hence $\text{ind}(Z) = \infty$). The transfinite inductive dimension trind is the natural generalization of ind ; see [5].

Theorem 5.2 *Assuming \diamond , there is a locally connected HS continuum Z such that $\dim(Z) = 1$, $\text{trind}(Z) = \infty$, and Z has no nontrivial convergent sequences.*

To do this, we make sure that all perfect subsets are G_δ sets. Observe that by local connectedness, every non-empty closed G_δ contains a non-empty connected closed G_δ subset, which in our Z cannot be a singleton. So, no non-empty closed G_δ can have dimension 0.

Lemma 5.3 *Assume that X is compact, connected, and infinite, and all perfect subsets of X are G_δ sets. Assume also that $\chi(x, X) > \aleph_0$ for all $x \in X$, and that in X , every non-empty closed G_δ set contains a non-empty closed connected G_δ subset. Then $\text{trind}(X) = \infty$.*

Proof. We prove by induction on ordinals α that $\neg[\text{trind}(X) \leq \alpha]$ for all such X . This is obvious for $\alpha = 0$. Assume $\alpha > 0$ and the inductive hypothesis holds for all ordinals $\xi < \alpha$. Suppose that $\text{trind}(X) \leq \alpha$. Then there is a regular open set U such that $U \neq \emptyset$, $U \neq X$, and $\text{trind}(\partial U) = \xi < \alpha$. Let $V = X \setminus \overline{U}$; then \overline{U} and \overline{V} are perfect, so $\partial U = \overline{U} \cap \overline{V}$ is a G_δ , and hence contains a non-empty closed connected G_δ subset Y . Then $\text{trind}(Y) \leq \text{trind}(\partial U) \leq \xi$. Since Y satisfies the conditions of the lemma, this is a contradiction. ☺

By the same argument, this space is *weird* in the sense of [8]; that is, no perfect subset is totally disconnected.

To construct our Z so that perfect sets are G_δ , we observe first that if $Q \subseteq \text{MS} \times [0, 1]^{\omega_1}$ is perfect, then $C := \{\alpha < \omega_1 : \pi_\alpha^{\omega_1}(Q) \text{ is perfect}\}$ is a club. One might then use \diamond , as in [4], to capture perfect subsets of Z , but this is not necessary, since we already know that Z is HS, and we are already capturing countable sequences. Thus, we get:

Conditions 5.4 *We have P_α and \mathcal{P}_α for $\alpha < \omega_1$ such that:*

C11. $P_\alpha = \text{cl}(Z_\alpha \cap \{q_\alpha^n : n \in \omega\})$ whenever $\alpha \geq \omega$ and this set is perfect; otherwise, $P_\alpha = Z_\alpha$.

C12. $\mathcal{P}_\alpha = \{(\sigma_\delta^\alpha)^{-1}(P_\delta) : \delta \leq \alpha\}$.

C13. $\sigma_\alpha^{\alpha+1} \upharpoonright ((\sigma_\alpha^{\alpha+1})^{-1}(P)) : (\sigma_\alpha^{\alpha+1})^{-1}(P) \twoheadrightarrow P$ is irreducible for each $P \in \mathcal{P}_\alpha$.

Proof of Theorem 5.2. To obtain these conditions, note that (C13) is trivial for P unless $t_\alpha \in P$. If $t_\alpha \in P$, then, since P is perfect, we may choose a sequence of distinct points $\langle p_n : n \in \omega \rangle$ from $P \setminus \{t_\alpha\}$ converging to t_α . Then, while we are accomplishing (C8), we make sure that all points of $(\sigma_\alpha^{\alpha+1})^{-1}\{t_\alpha\}$ are (strong) limit points of the set of singletons, $\{(\sigma_\alpha^{\alpha+1})^{-1}\{p_n\} : n \in \omega\}$; this implies irreducibility.

Now, we prove by induction on $\beta \geq \alpha$ that $\sigma_\alpha^\beta \upharpoonright ((\sigma_\alpha^\beta)^{-1}(P)) : (\sigma_\alpha^\beta)^{-1}(P) \twoheadrightarrow P$ is irreducible for each $P \in \mathcal{P}_\alpha$. Then, if $Q \subseteq Z$ is perfect, we use HS and (C7) to fix some $\alpha < \omega_1$ such that $P_\alpha = \sigma_\alpha^{\omega_1}(Q)$ and P_α is perfect. Irreducibility then implies that $Q = (\sigma_\alpha^{\omega_1})^{-1}(P_\alpha)$, which is a G_δ . ☺

Finally, we remark that our space Z is *dissipated* in the sense of [10], since in the inverse limit, only one point t_α gets expanded in passing from Z_α to $Z_{\alpha+1}$; the inverse projection of every other point is a singleton. As pointed out in [10], this is also true of the original Fedorchuk S-space [6], where one point t_α got expanded to a pair of points; here, and in [8] and van Mill [13], t_α gets expanded to an interval.

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